

# On the average value of the least common multiple of $k$ positive integers

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## Abstract

We deduce an asymptotic formula with error term for the sum  $\sum_{n_1, \dots, n_k \leq x} f([n_1, \dots, n_k])$ , where  $[n_1, \dots, n_k]$  stands for the least common multiple of the positive integers  $n_1, \dots, n_k$  ( $k \geq 2$ ) and  $f$  belongs to a large class of multiplicative arithmetic functions, including, among others, the functions  $f(n) = n^r$ ,  $\varphi(n)^r$ ,  $\sigma(n)^r$  ( $r > -1$  real), where  $\varphi$  is Euler's totient function and  $\sigma$  is the sum-of-divisors function. The proof is by elementary arguments, using the extension of the convolution method for arithmetic functions of several variables, starting with the observation that given a multiplicative function  $f$ , the function of  $k$  variables  $f([n_1, \dots, n_k])$  is multiplicative.

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## 1 Introduction

We use the following notation:  $\mathbb{N} = \{1, 2, \dots\}$ ,  $*$  is the Dirichlet convolution of arithmetic functions,  $\text{id}_r$  ( $r \in \mathbb{R}$ ) is the function  $\text{id}_r(n) = n^r$  ( $n \in \mathbb{N}$ ),  $\mathbf{1} = \text{id}_0$ ,  $\text{id} = \text{id}_1$ ,  $\mu$  denotes the Möbius function,  $\lambda$  is the Liouville function,  $\sigma_r = \mathbf{1} * \text{id}_r$ ,  $\sigma = \sigma_1$  is the sum-of-divisors function,  $\tau = \sigma_0$  is the divisor function,  $\beta_r = \lambda * \text{id}_r$ ,  $\beta = \beta_1$  is the alternating sum-of-divisors function (cf. [19]),  $\varphi_r = \mu * \text{id}_r$  is the generalized Euler function,  $\varphi = \varphi_1$  is Euler's totient function,  $\psi_r = \mu^2 * \text{id}_r$  is the generalized Dedekind function,  $\psi = \psi_1$  is the classical Dedekind function. If  $n \in \mathbb{N}$ , then  $n = \prod_p p^{\nu_p(n)}$  is its prime power factorization, the product being over the primes  $p$ , where all but a finite number of the exponents  $\nu_p(n)$  are zero.

Furthermore, let  $(n_1, \dots, n_k)$  and  $[n_1, \dots, n_k]$  denote the greatest common divisor (gcd) and the least common multiple (lcm) of  $n_1, \dots, n_k \in \mathbb{N}$  ( $k \geq 2$ ), respectively.

It is easy to see that for any arithmetic function  $f$  we have the identity

$$\sum_{n_1, \dots, n_k \leq x} f([n_1, \dots, n_k]) = \sum_{d \leq x} (\mu * f)(d) \left\lfloor \frac{x}{d} \right\rfloor^k, \quad (1)$$

which leads to asymptotic formulas for this sum. For example, if  $f = \text{id}$  and  $k \geq 3$ , then we have

$$\sum_{n_1, \dots, n_k \leq x} (n_1, \dots, n_k) = \frac{\zeta(k-1)}{\zeta(k)} x^k + O(R_k(x)), \quad (2)$$

where  $R_3(x) = x^2 \log x$  and  $R_k(x) = x^{k-1}$  for  $k \geq 4$ . The case  $f = \text{id}$ ,  $k = 2$  can be treated separately by writing

$$\begin{aligned} \sum_{m,n \leq x} (m, n) &= 2 \sum_{m \leq n \leq x} (m, n) - \sum_{n \leq x} n \\ &= 2 \sum_{n \leq x} (\mu * \text{id } \tau)(n) - \frac{x^2}{2} + O(x), \end{aligned}$$

giving, by using elementary arguments, the formula

$$\sum_{m,n \leq x} (m, n) = \frac{x^2}{\zeta(2)} \left( \log x + 2\gamma - \frac{1}{2} - \frac{\zeta(2)}{2} - \frac{\zeta'(2)}{\zeta(2)} \right) + O(x^{1+\theta+\varepsilon}), \quad (3)$$

valid for every  $\varepsilon > 0$ , where  $\gamma$  is Euler's constant and  $\theta$  is the exponent appearing in Dirichlet's divisor problem.

For the lcm of  $k$  positive integers there is no formula similar to (1). However, in the case  $k = 2$ , the lcm of the integers  $m, n \in \mathbb{N}$  can be written using their gcd as  $[m, n] = mn/(m, n)$ , which enables to establish the following asymptotic formula, valid for any positive real number  $r$ :

$$\sum_{m,n \leq x} [m, n]^r = \frac{\zeta(r+2)}{\zeta(2)} \cdot \frac{x^{2(r+1)}}{(r+1)^2} + O(x^{2r+1} \log x). \quad (4)$$

If  $r \in \mathbb{N}$ , then the error term in (4) can be improved into  $O(x^{2r+1}(\log x)^{2/3}(\log \log x)^{4/3})$ , which is a consequence of the result of Walfisz [23, Satz 1, p. 144] for  $\sum_{n \leq x} \varphi(n)$ .

For  $k = 2$  the asymptotic formulas concerning  $\sum_{m,n \leq x} (m, n)^r$  and  $\sum_{m,n \leq x} [m, n]^r$  are equivalent to those for  $\sum_{n \leq x} g_r(n)$  and  $\sum_{n \leq x} \ell_r(n)$ , respectively, where  $g_r(n) = \sum_{1 \leq j \leq n} (j, n)^r$  is the gcd-sum function and  $\ell_r(n) = \sum_{1 \leq j \leq n} [j, n]^r$  is the lcm-sum function. The function  $g_1(n) = \sum_{1 \leq j \leq n} (j, n)$ , investigated by S. S. Pillai [16], is also called Pillai's function in the literature.

The above and related results go back, in chronological order, to the work of E. Cesàro [6], E. Cohen [9, 10, 11], K. Alladi [1], P. Diaconis and P. Erdős [12], J. Chidambaraswamy and R. Sitaramachandrarao [7], K. A. Broughan [5], O. Bordellès [2, 3, 4], Y. Tanigawa and W. Zhai [17], S. Ikeda and K. Matsuoka [15], and others.

For example, formula (3) with the weaker error  $O(x^{3/2} \log x)$  was given in [12, Th. 2, Eq. (1.4)] and was recovered in [5, Th. 4.7]. Formula (3) with the above error term was established in [7, Th. 3.1] and recovered in [2, Th. 1.1] (in both papers for Pillai's function). Formula (4) was established in [12, Th. 2, Eq. (1.6)]. The better error term for (4) in the case  $r \in \mathbb{N}$  was obtained in [15, Th. 2]. Asymptotic formulas for (1) in the case  $k = 2$  and for various choices of the function  $f$ , including  $f = \sigma$  and  $f = \varphi$  were deduced in [4, 9, 10, 11]. See also the survey paper [18].

The result

$$\sum_{m,n,q \leq x} [m, n, q]^r \sim c_r \frac{x^{3(r+1)}}{(r+1)^3} \quad (x \rightarrow \infty),$$

valid for  $r \in \mathbb{N}$ , without any error term and with a computable constant  $c_r$  given in an implicit form, was obtained by J. L. Fernández and P. Fernández [13, Th. 3(b)]. Their proof is by an ingenious method based on the identity  $[m, n, q](m, n)(m, q)(n, q) = mnq(m, n, q)$  ( $m, n, q \in \mathbb{N}$ ) and using the dominated convergence theorem. As far as we know, there are no other asymptotic results in the literature for the sum

$$\sum_{n_1, \dots, n_k \leq x} f([n_1, \dots, n_k]), \quad (5)$$

in the case  $k \geq 3$ , where  $f$  is an arithmetic function. It seems that the method of [13] can not be extended for  $k \geq 3$ , even in the case  $f = \text{id}_r$ . Also, it is not possible to reduce the estimation of the sum (5) to sums of a single variable, like in (1).

In this paper we deduce an asymptotic formula with remainder term for the sum (5), where  $k \geq 2$  and  $f$  belongs to a large class of multiplicative arithmetic functions, including the functions  $\text{id}_r$  with  $r > -1$  real and  $\sigma_r, \beta_r, \varphi_r, \psi_r$  with  $r \geq 1/2$  real. The proof is by elementary arguments, using the extension of the convolution method for arithmetic functions of several variables starting with the observation that given a multiplicative function  $f$ , the function of  $k$  variables  $f([n_1, \dots, n_k])$  is multiplicative and the associated multiple Dirichlet series factorizes as an Euler product. The same method was used by the second author [?] for a different problem. See the survey paper [20] of the second author for basic properties of multiplicative functions of several variables and related convolutions.

We also extend to the  $k$  dimensional case the formula

$$\sum_{m, n \leq x} \frac{[m, n]}{(m, n)} = \frac{\pi^2}{60} x^4 + O(x^3 \log x), \quad (6)$$

which can be obtained in a similar manner to the results (2) and (4). Properties of the operation  $m \circ n = [m, n]/(m, n)$  were investigated by the first author [14].

Note that the following recent result of different type, concerning the lcm of several positive integers, was obtained by J. Cilleruelo, J. Rué, P. Šarka and A. Zumalacárregui [8]:  $\text{lcm}\{a : a \in A\} = 2^{n(1+o(1))}$  for almost all subsets  $A \subset \{1, \dots, n\}$ .

## 2 Main results

Let  $r \in \mathbb{R}$  be a fixed number. Let  $\mathcal{A}_r$  denote the class of complex valued multiplicative arithmetic functions satisfying the following properties: there exist real constants  $C_1, C_2$  such that

$$|f(p) - p^r| \leq C_1 p^{r-1/2} \quad \text{for every prime } p, \quad (i)$$

and

$$|f(p^\nu)| \leq C_2 p^{\nu r} \quad \text{for every prime power } p^\nu \text{ with } \nu \geq 2. \quad (ii)$$

Note that conditions (i) and (ii) imply that

$$|f(p^\nu)| \leq C_3 p^{\nu r} \quad \text{for every prime power } p^\nu \text{ with } \nu \geq 1, \quad (iii)$$

where  $C_3 = \max(C_1 + 1, C_2)$ .

Observe that  $\text{id}_r \in \mathcal{A}_r$  for every  $r \in \mathbb{R}$ , while  $\sigma_r, \beta_r, \varphi_r, \psi_r \in \mathcal{A}_r$  for every  $r \in \mathbb{R}$  with  $r \geq 1/2$ . The functions  $f(n) = \sigma(n)^r, \beta(n)^r, \varphi(n)^r, \psi(n)^r$  also belong to the class  $\mathcal{A}_r$  for every  $r \in \mathbb{R}$ . As other examples of functions in the class  $\mathcal{A}_r$ , with  $r \in \mathbb{R}$ , we mention  $\varphi^*(n)^r, \sigma^*(n)^r$  and  $\sigma^{(e)}(n)^r$ , where  $\varphi^*(n) = \prod_{p|n} (p^{\nu_p(n)} - 1)$  is the unitary Euler totient,  $\sigma^*(n) = \prod_{p|n} (p^{\nu_p(n)} + 1)$  is the sum-of-unitary-divisors function and  $\sigma^{(e)}(n) = \prod_{p|n} \sum_{d|\nu_p(n)} p^d$  denotes the sum of exponential divisors of  $n$ . Furthermore, if  $f$  is a bounded multiplicative function such that  $f(p) = 1$  for every prime  $p$ , then  $f \in \mathcal{A}_0$ . In particular,  $\mu^2 \in \mathcal{A}_0$ .

We prove the following results.

**Theorem 2.1.** Let  $k \geq 2$  be a fixed integer and let  $f \in \mathcal{A}_r$  be a function, where  $r > -1$  is real. Then for every  $\varepsilon > 0$ ,

$$\sum_{n_1, \dots, n_k \leq x} f([n_1, \dots, n_k]) = C_{f,k} \frac{x^{k(r+1)}}{(r+1)^k} + O\left(x^{k(r+1) - \frac{1}{2} \min(r+1, 1) + \varepsilon}\right), \quad (7)$$

and

$$\sum_{n_1, \dots, n_k \leq x} \frac{f([n_1, \dots, n_k])}{(n_1 \cdots n_k)^r} = C_{f,k} x^k + O\left(x^{k - \frac{1}{2} \min(r+1, 1) + \varepsilon}\right), \quad (8)$$

where

$$C_{f,k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1, \dots, \nu_k=0}^{\infty} \frac{f(p^{\max(\nu_1, \dots, \nu_k)})}{p^{(r+1)(\nu_1 + \dots + \nu_k)}}.$$

Formula (7) shows that the average order of  $f([n_1, \dots, n_k])$  is  $C_{f,k}(n_1 \cdots n_k)^r$ , in the sense that

$$\sum_{n_1, \dots, n_k \leq x} f([n_1, \dots, n_k]) \sim \sum_{n_1, \dots, n_k \leq x} C_{f,k}(n_1 \cdots n_k)^r \quad (x \rightarrow \infty).$$

From (8) we deduce that

$$\lim_{x \rightarrow \infty} \frac{1}{x^k} \sum_{n_1, \dots, n_k \leq x} \frac{f([n_1, \dots, n_k])}{(n_1 \cdots n_k)^r} = C_{f,k},$$

representing the mean value of the function  $f([n_1, \dots, n_k])/(n_1 \cdots n_k)^r$ . See N. Ushiroya [22, Th. 4] and the second author [20, Prop. 19] for general results on mean values of multiplicative arithmetic functions of several variables.

**Theorem 2.2.** Let  $k \geq 2$  be a fixed integer and let  $f \in \mathcal{A}_r$  be a function, where  $r \geq 0$  is real. Then for every  $\varepsilon > 0$ ,

$$\sum_{n_1, \dots, n_k \leq x} f\left(\frac{[n_1, \dots, n_k]}{(n_1, \dots, n_k)}\right) = D_{f,k} \frac{x^{k(r+1)}}{(r+1)^k} + O\left(x^{k(r+1) - \frac{1}{2} + \varepsilon}\right), \quad (9)$$

where

$$D_{f,k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1, \dots, \nu_k=0}^{\infty} \frac{f(p^{\max(\nu_1, \dots, \nu_k) - \min(\nu_1, \dots, \nu_k)})}{p^{(r+1)(\nu_1 + \dots + \nu_k)}}.$$

In the case  $f = \text{id}_r$  we obtain from Theorem 2.1 the next result:

**Corollary 1.** Let  $k \geq 3$  and  $r > -1$  be a real number. Then for every  $\varepsilon > 0$ ,

$$\sum_{n_1, \dots, n_k \leq x} [n_1, \dots, n_k]^r = C_{r,k} \frac{x^{k(r+1)}}{(r+1)^k} + O\left(x^{k(r+1) - \frac{1}{2} \min(r+1, 1) + \varepsilon}\right), \quad (10)$$

and

$$\sum_{n_1, \dots, n_k \leq x} \left(\frac{[n_1, \dots, n_k]}{n_1 \cdots n_k}\right)^r = C_{r,k} x^k + O\left(x^{k - \frac{1}{2} \min(r+1, 1) + \varepsilon}\right),$$

where

$$C_{r,k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1, \dots, \nu_k=0}^{\infty} \frac{p^{r \max(\nu_1, \dots, \nu_k)}}{p^{(r+1)(\nu_1 + \dots + \nu_k)}}.$$

In particular,

$$C_{r,3} = \zeta(r+2)\zeta(2r+3) \prod_p \left(1 - \frac{3}{p^2} + \frac{2}{p^3} + \frac{2}{p^{r+2}} - \frac{3}{p^{r+3}} + \frac{1}{p^{r+5}}\right), \quad (11)$$

$$C_{r,4} = \zeta(r+2)\zeta(2r+3)\zeta(3r+4) \prod_p \left(1 - \frac{6}{p^2} + \frac{8}{p^3} - \frac{3}{p^4} + \frac{5}{p^{r+2}} - \frac{12}{p^{r+3}} + \frac{6}{p^{r+4}} + \frac{4}{p^{r+5}} - \frac{3}{p^{r+6}} + \frac{3}{p^{2r+3}} - \frac{4}{p^{2r+4}} - \frac{6}{p^{2r+5}} + \frac{12}{p^{2r+6}} - \frac{5}{p^{2r+7}} + \frac{3}{p^{3r+5}} - \frac{8}{p^{3r+6}} + \frac{6}{p^{3r+7}} - \frac{1}{p^{3r+9}}\right). \quad (12)$$

In the case  $f = \text{id}_r$  we deduce from Theorem 2.2:

**Corollary 2.** *Let  $k \geq 3$  and  $r > 0$  be a real number. Then for every  $\varepsilon > 0$ ,*

$$\sum_{n_1, \dots, n_k \leq x} \left( \frac{[n_1, \dots, n_k]}{(n_1, \dots, n_k)} \right)^r = D_{r,k} \frac{x^{k(r+1)}}{(r+1)^k} + O\left(x^{k(r+1)-\frac{1}{2}+\varepsilon}\right), \quad (13)$$

where

$$D_{r,k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1, \dots, \nu_k=0}^{\infty} \frac{p^{r(\max(\nu_1, \dots, \nu_k) - \min(\nu_1, \dots, \nu_k))}}{p^{(r+1)(\nu_1 + \dots + \nu_k)}}.$$

In particular,

$$D_{r,3} = C_{r,3} \frac{\zeta(3r+3)}{\zeta(2r+3)}, \quad D_{r,4} = C_{r,4} \frac{\zeta(4r+4)}{\zeta(3r+4)}.$$

We remark that in the case  $k = 2$  asymptotic formulas (10) and (13) reduce to (4) and (6) (case  $r = 1$ ), respectively, but the latter ones have better error terms. Note that  $D_{r,2} = \zeta(2r+2)/\zeta(2)$ .

Among other special cases we consider the functions  $\sigma, \varphi \in \mathcal{A}_1$  and  $\mu^2 \in \mathcal{A}_0$ .

**Corollary 3.** *Let  $k \geq 2$ . Then for every  $\varepsilon > 0$ ,*

$$\sum_{n_1, \dots, n_k \leq x} \sigma([n_1, \dots, n_k]) = C_{\sigma,k} \frac{x^{2k}}{2^k} + O\left(x^{2k-1/2+\varepsilon}\right),$$

and

$$\sum_{n_1, \dots, n_k \leq x} \frac{\sigma([n_1, \dots, n_k])}{n_1 \cdots n_k} = C_{\sigma,k} x^k + O\left(x^{k-1/2+\varepsilon}\right),$$

where

$$C_{\sigma,k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1, \dots, \nu_k=0}^{\infty} \frac{\sigma(p^{\max(\nu_1, \dots, \nu_k)})}{p^{2(\nu_1 + \dots + \nu_k)}}.$$

In particular,

$$C_{\sigma,2} = \zeta(3)\zeta(4) \prod_p \left(1 + \frac{1}{p^2} - \frac{2}{p^3} - \frac{2}{p^5} + \frac{2}{p^6}\right).$$

**Corollary 4.** *Let  $k \geq 2$ . Then for every  $\varepsilon > 0$ ,*

$$\sum_{n_1, \dots, n_k \leq x} \varphi([n_1, \dots, n_k]) = C_{\varphi, k} \frac{x^{2k}}{2^k} + O\left(x^{2k-1/2+\varepsilon}\right),$$

and

$$\sum_{n_1, \dots, n_k \leq x} \frac{\varphi([n_1, \dots, n_k])}{n_1 \cdots n_k} = C_{\varphi, k} x^k + O\left(x^{k-1/2+\varepsilon}\right),$$

where

$$C_{\varphi, k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1, \dots, \nu_k=0}^{\infty} \frac{\varphi(p^{\max(\nu_1, \dots, \nu_k)})}{p^{2(\nu_1 + \dots + \nu_k)}}.$$

In particular,

$$C_{\varphi, 2} = \zeta(3) \prod_p \left(1 - \frac{3}{p^2} + \frac{2}{p^3} - \frac{1}{p^4} + \frac{2}{p^5} - \frac{1}{p^6}\right).$$

**Corollary 5.** *Let  $k \geq 2$ . Then for every  $\varepsilon > 0$ ,*

$$\sum_{n_1, \dots, n_k \leq x} \mu^2([n_1, \dots, n_k]) = \frac{x^k}{\zeta(2)^k} + O\left(x^{k-1/2+\varepsilon}\right).$$

**Remark 1.** It would be interesting to find the best possible error, especially in particular cases. For example, for  $r = 1$  in Corollary 1, the relative error is  $O(x^{-1/2+\varepsilon})$ . Can we improve the exponent further and if so, by how much?

### 3 Proofs

An arithmetic function  $g$  of  $k$  variables is called multiplicative if it is not identically zero and

$$g(m_1 n_1, \dots, m_k n_k) = g(m_1, \dots, m_k) g(n_1, \dots, n_k),$$

provided that  $(m_1 \cdots m_k, n_1 \cdots n_k) = 1$ . Hence

$$g(n_1, \dots, n_k) = \prod_p g\left(p^{\nu_p(n_1)}, \dots, p^{\nu_p(n_k)}\right)$$

for every  $n_1, \dots, n_k \in \mathbb{N}$ . In this case the multiple Dirichlet series of the function  $g$  can be expanded into an Euler product:

$$\sum_{n_1, \dots, n_k=1}^{\infty} \frac{g(n_1, \dots, n_k)}{n_1^{z_1} \cdots n_k^{z_k}} = \prod_p \sum_{\nu_1, \dots, \nu_k=0}^{\infty} \frac{g(p^{\nu_1}, \dots, p^{\nu_k})}{p^{\nu_1 z_1 + \dots + \nu_k z_k}}.$$

We need the following lemmas.

**Lemma 3.1.** *If  $k \geq 2$  and  $f \in \mathcal{A}_r$  with  $r > -1$  real, then*

$$L_{f, k}(z_1, \dots, z_k) := \sum_{n_1, \dots, n_k=1}^{\infty} \frac{f([n_1, \dots, n_k])}{n_1^{z_1} \cdots n_k^{z_k}} = \zeta(z_1 - r) \cdots \zeta(z_k - r) H_{f, k}(z_1, \dots, z_k),$$

where the multiple Dirichlet series  $H_{f, k}(z_1, \dots, z_k)$  is absolutely convergent for

$$\Re z_1, \dots, \Re z_k > A := \begin{cases} r + \frac{1}{2}, & \text{if } r \geq 0, \\ \frac{r+1}{2}, & \text{if } -1 < r < 0. \end{cases} \quad (14)$$

*Proof.* If  $f$  is a multiplicative function of a single variable, then the arithmetic function of  $k$  variables  $f([n_1, \dots, n_k])$  is multiplicative. It follows that

$$L_{f,k}(z_1, \dots, z_k) = \prod_p \sum_{\nu_1, \dots, \nu_k=0}^{\infty} \frac{f(p^{\max(\nu_1, \dots, \nu_k)})}{p^{\nu_1 z_1 + \dots + \nu_k z_k}} \quad (15)$$

Case I. Assume that  $r \geq 0$ . Grouping the terms of the sum in (15) according to the values  $\nu_1 + \dots + \nu_k$  we have

$$L_{f,k}(z_1, \dots, z_k) = \prod_p \left( 1 + \frac{f(p)}{p^{z_1}} + \dots + \frac{f(p)}{p^{z_k}} + \sum_{\nu_1 + \dots + \nu_k \geq 2} \frac{f(p^{\max(\nu_1, \dots, \nu_k)})}{p^{\nu_1 z_1 + \dots + \nu_k z_k}} \right). \quad (16)$$

Let  $\Re z_1, \dots, \Re z_k \geq \delta > r$ . By using condition (i) from the definition of the class  $\mathcal{A}_r$ ,

$$\frac{f(p)}{p^{z_j}} = \frac{1}{p^{z_j - r}} + O\left(\frac{1}{p^{\delta - r + 1/2}}\right) \quad (1 \leq j \leq k).$$

Also, by condition (iii) following the definition of the class  $\mathcal{A}_r$  and by using that  $r \geq 0$  we deduce that

$$\left| \frac{f(p^{\max(\nu_1, \dots, \nu_k)})}{p^{\nu_1 z_1 + \dots + \nu_k z_k}} \right| \leq C_3 \frac{p^{r \max(\nu_1, \dots, \nu_k)}}{p^{\delta(\nu_1 + \dots + \nu_k)}} \leq C_3 \frac{1}{p^{(\delta - r)(\nu_1 + \dots + \nu_k)}}.$$

Thus the sum in (16) over  $\nu_1 + \dots + \nu_k \geq 2$  is  $O(p^{-2(\delta - r)})$ . We obtain

$$\begin{aligned} & L_{f,k}(z_1, \dots, z_k) \zeta^{-1}(z_1 - r) \dots \zeta^{-1}(z_k - r) \\ &= \prod_p \left( 1 - \frac{1}{p^{z_1 - r}} \right) \dots \left( 1 - \frac{1}{p^{z_k - r}} \right) \left( 1 + \frac{1}{p^{z_1 - r}} + \dots + \frac{1}{p^{z_k - r}} + O\left(\frac{1}{p^{\delta - r + 1/2}}\right) \right. \\ & \quad \left. + O\left(\frac{1}{p^{2(\delta - r)}}\right) \right) = \prod_p \left( 1 + O\left(\frac{1}{p^{\delta - r + 1/2}}\right) + O\left(\frac{1}{p^{2(\delta - r)}}\right) \right), \end{aligned}$$

since  $\Re z_j \geq \delta$  ( $1 \leq j \leq k$ ), where the terms  $\pm \frac{1}{p^{z_j - r}}$  ( $1 \leq j \leq k$ ) cancel out. Here the latter product converges absolutely when  $\delta - r + 1/2 > 1$  and  $2(\delta - r) > 1$ , that is, for  $\delta > r + 1/2$ .

Case II. Assume that  $-1 < r < 0$ . Now we group the terms of the sum in (15) according to the values  $\max(\nu_1, \dots, \nu_k)$ :

$$L_{f,k}(z_1, \dots, z_k) = \prod_p \left( 1 + \sum_{\max(\nu_1, \dots, \nu_k)=1} \frac{f(p)}{p^{\nu_1 z_1 + \dots + \nu_k z_k}} + \sum_{\max(\nu_1, \dots, \nu_k) \geq 2} \frac{f(p^{\max(\nu_1, \dots, \nu_k)})}{p^{\nu_1 z_1 + \dots + \nu_k z_k}} \right). \quad (17)$$

Let  $\Re z_1, \dots, \Re z_k \geq \delta \geq 0$ . Consider the sum in (17) over  $\max(\nu_1, \dots, \nu_k) = 1$  and suppose that  $\nu_i = 1$  for  $m$  ( $1 \leq m \leq k$ ) distinct values of  $i$ . If  $m = 1$ , then by condition (i) from the definition of the class  $\mathcal{A}_r$  we have

$$\frac{f(p)}{p^{z_j}} = \frac{1}{p^{z_j - r}} + O\left(\frac{1}{p^{\delta - r + 1/2}}\right) \quad (1 \leq j \leq k).$$

If  $m \geq 2$ , then

$$\left| \frac{f(p)}{p^{\nu_1 z_1 + \dots + \nu_k z_k}} \right| \leq \frac{(C_1 + 1)p^r}{p^{m\delta}} = O\left(\frac{1}{p^{2\delta - r}}\right).$$

This shows that the sum in (17) over  $\max(\nu_1, \dots, \nu_k) = 1$  is

$$\frac{1}{p^{z_1-r}} + \dots + \frac{1}{p^{z_k-r}} + O\left(\frac{1}{p^{\delta-r+1/2}}\right) + O\left(\frac{1}{p^{2\delta-r}}\right).$$

Furthermore, by condition (ii) we deduce that for  $\max(\nu_1, \dots, \nu_k) \geq 2$ ,

$$\left| \frac{f(p^{\max(\nu_1, \dots, \nu_k)})}{p^{\nu_1 z_1 + \dots + \nu_k z_k}} \right| \leq C_2 \frac{p^{r \max(\nu_1, \dots, \nu_k)}}{p^{\delta(\nu_1 + \dots + \nu_k)}} \leq C_2 \frac{1}{p^{(\delta-r) \max(\nu_1, \dots, \nu_k)}}$$

( $\delta \geq 0$ ) and it follows that the sum in (17) over  $\max(\nu_1, \dots, \nu_k) \geq 2$  is  $O(p^{-2(\delta-r)}) = O(p^{-(2\delta-r)})$ , since  $r < 0$ .

We obtain that

$$L_{f,k}(z_1, \dots, z_k) = \prod_p \left( 1 + \frac{1}{p^{z_1-r}} + \dots + \frac{1}{p^{z_k-r}} + O\left(\frac{1}{p^{\delta-r+1/2}}\right) + O\left(\frac{1}{p^{2\delta-r}}\right) \right)$$

and

$$\begin{aligned} & L_{f,k}(z_1, \dots, z_k) \zeta^{-1}(z_1-r) \dots \zeta^{-1}(z_k-r) \\ &= \prod_p \left( 1 - \frac{1}{p^{z_1-r}} \right) \dots \left( 1 - \frac{1}{p^{z_k-r}} \right) \prod_p \left( 1 + \frac{1}{p^{z_1-r}} + \dots + \frac{1}{p^{z_k-r}} \right. \\ & \quad \left. + O\left(\frac{1}{p^{\delta-r+1/2}}\right) + O\left(\frac{1}{p^{2\delta-r}}\right) \right) \\ &= \prod_p \left( 1 + O\left(\frac{1}{p^{\delta-r+1/2}}\right) + O\left(\frac{1}{p^{2\delta-r}}\right) \right), \end{aligned}$$

since  $\Re z_j \geq \delta$  ( $1 \leq j \leq k$ ), where the terms  $\pm \frac{1}{p^{z_j-r}}$  ( $1 \leq j \leq k$ ) cancel out, similar to Case I. Here the latter product converges absolutely when  $\delta - r + 1/2 > 1$  and  $2\delta - r > 1$ , that is, for  $\delta > (r+1)/2 > 0$ .  $\square$

**Lemma 3.2.** *If  $k \geq 2$  and  $f \in \mathcal{A}_r$  with  $r \geq 0$ , then*

$$\bar{L}_{f,k}(z_1, \dots, z_k) := \sum_{n_1, \dots, n_k=1}^{\infty} \frac{f\left(\frac{[n_1, \dots, n_k]}{(n_1, \dots, n_k)}\right)}{n_1^{z_1} \dots n_k^{z_k}} = \zeta(z_1-r) \dots \zeta(z_k-r) \bar{H}_{f,k}(z_1, \dots, z_k),$$

where the multiple Dirichlet series  $\bar{H}_{f,k}(z_1, \dots, z_k)$  is absolutely convergent for  $\Re z_1, \dots, \Re z_k > r + 1/2$ .

*Proof.* Similar to the proof of Lemma 3.1, Case I. If  $f$  is multiplicative, then the function  $f([n_1, \dots, n_k]/(n_1, \dots, n_k))$  is also multiplicative and we have

$$\begin{aligned} \bar{L}_{f,k}(z_1, \dots, z_k) &= \prod_p \sum_{\nu_1, \dots, \nu_k=0}^{\infty} \frac{f(p^{\max(\nu_1, \dots, \nu_k) - \min(\nu_1, \dots, \nu_k)})}{p^{\nu_1 z_1 + \dots + \nu_k z_k}} \\ &= \prod_p \left( 1 + \frac{f(p)}{p^{z_1}} + \dots + \frac{f(p)}{p^{z_k}} + \sum_{\nu_1 + \dots + \nu_k \geq 2} \frac{f(p^{\max(\nu_1, \dots, \nu_k) - \min(\nu_1, \dots, \nu_k)})}{p^{\nu_1 z_1 + \dots + \nu_k z_k}} \right). \end{aligned} \quad (18)$$



If  $\Re z_1, \dots, \Re z_k \geq \delta > r$ , then it follows that

$$\left| \frac{f(p^{\max(\nu_1, \dots, \nu_k) - \min(\nu_1, \dots, \nu_k)})}{p^{\nu_1 z_1 + \dots + \nu_k z_k}} \right| \leq C \frac{p^{r(\max(\nu_1, \dots, \nu_k) - \min(\nu_1, \dots, \nu_k))}}{p^{\delta(\nu_1 + \dots + \nu_k)}} \leq C \frac{1}{p^{(\delta-r)(\nu_1 + \dots + \nu_k)}},$$

thus the sum in (18) over  $\nu_1 + \dots + \nu_k \geq 2$  is  $O(p^{-2(\delta-r)})$ . Furthermore, we use the same arguments as in the previous proof.  $\square$

*Proof of Theorem 2.1.* From Lemma 3.1 we deduce the convolutional identity

$$f([n_1, \dots, n_k]) = \sum_{j_1 d_1 = n_1, \dots, j_k d_k = n_k} j_1^r \cdots j_k^r h_{f,k}(d_1, \dots, d_k),$$

where

$$\sum_{n_1, \dots, n_k=1}^{\infty} \frac{h_{f,k}(n_1, \dots, n_k)}{n_1^{z_1} \cdots n_k^{z_k}} = H_{f,k}(z_1, \dots, z_k).$$

Therefore

$$\begin{aligned} \sum_{n_1, \dots, n_k \leq x} f([n_1, \dots, n_k]) &= \sum_{j_1 d_1 \leq x, \dots, j_k d_k \leq x} j_1^r \cdots j_k^r h_{f,k}(d_1, \dots, d_k) \\ &= \sum_{d_1, \dots, d_k \leq x} h_{f,k}(d_1, \dots, d_k) \sum_{j_1 \leq x/d_1} j_1^r \cdots \sum_{j_k \leq x/d_k} j_k^r \\ &= \sum_{d_1, \dots, d_k \leq x} h_{f,k}(d_1, \dots, d_k) \left( \frac{x^{r+1}}{(r+1)d_1^{r+1}} + O\left(\frac{x^R}{d_1^R}\right) \right) \cdots \left( \frac{x^{r+1}}{(r+1)d_k^{r+1}} + O\left(\frac{x^R}{d_k^R}\right) \right), \end{aligned}$$

where  $R := \max(r, 0)$ . We deduce that

$$\sum_{n_1, \dots, n_k \leq x} f([n_1, \dots, n_k]) = \frac{x^{k(r+1)}}{(r+1)^k} \sum_{d_1, \dots, d_k \leq x} \frac{h_{f,k}(d_1, \dots, d_k)}{d_1^{r+1} \cdots d_k^{r+1}} + S_{k,r}(x), \quad (19)$$

with

$$S_{k,r}(x) \ll \sum_{u_1, \dots, u_k} x^{u_1 + \dots + u_k} \sum_{d_1, \dots, d_k \leq x} \frac{|h_{f,k}(d_1, \dots, d_k)|}{d_1^{u_1} \cdots d_k^{u_k}}, \quad (20)$$

where the first sum is over  $u_1, \dots, u_k \in \{r+1, R\}$  such that at least one  $u_i$  is  $R$ . Let  $u_1, \dots, u_k$  be fixed and assume that  $u_i = R$  for  $t$  ( $1 \leq t \leq k$ ) values of  $i$ , we take the first  $t$  values of  $i$ . Then  $x^{u_1 + \dots + u_k}$  times the inner sum of (20) is, using the notation  $A$  given by (14),

$$\begin{aligned} &\ll x^{(k-t)(r+1)+tR} \sum_{d_1, \dots, d_k \leq x} \frac{|h_{f,k}(d_1, \dots, d_k)|}{d_1^R \cdots d_t^R d_{t+1}^{r+1} \cdots d_k^{r+1}} \\ &= x^{(k-t)(r+1)+tR} \sum_{d_1, \dots, d_k \leq x} \frac{|h_{f,k}(d_1, \dots, d_k)| d_1^{A-R+\varepsilon} \cdots d_t^{A-R+\varepsilon}}{d_1^{A+\varepsilon} \cdots d_t^{A+\varepsilon} d_{t+1}^{r+1} \cdots d_k^{r+1}} \\ &\leq x^{(k-t)(r+1)+tR} x^{t(A-R+\varepsilon)} \sum_{d_1, \dots, d_k=1}^{\infty} \frac{|h_{f,k}(d_1, \dots, d_k)|}{d_1^{A+\varepsilon} \cdots d_t^{A+\varepsilon} d_{t+1}^{r+1} \cdots d_k^{r+1}} \\ &= x^{k(r+1)-t(r+1-A)+t\varepsilon} H_{f,k}(A+\varepsilon, \dots, A+\varepsilon, r+1, \dots, r+1) \\ &\ll x^{k(r+1)-t(r+1-A)+t\varepsilon}, \end{aligned}$$

since the latter series is convergent by Lemma 3.1. Using that  $r+1-A = \frac{1}{2} \min(r+1, 1) > 0$ , the obtained error is maximal for  $t = 1$  giving

$$O\left(x^{k(r+1)-\frac{1}{2}\min(r+1,1)+\varepsilon}\right).$$

Furthermore, for the sum in the main term of (19) we have

$$\begin{aligned} & \sum_{d_1, \dots, d_k \leq x} \frac{h_{f,k}(d_1, \dots, d_k)}{d_1^{r+1} \dots d_k^{r+1}} \\ &= \sum_{d_1, \dots, d_k=1}^{\infty} \frac{h_{f,k}(d_1, \dots, d_k)}{d_1^{r+1} \dots d_k^{r+1}} - \sum_{\emptyset \neq I \subseteq \{1, \dots, k\}} \sum_{\substack{d_i > x, i \in I \\ d_j \leq x, j \notin I}} \frac{h_{f,k}(d_1, \dots, d_k)}{d_1^{r+1} \dots d_k^{r+1}}, \end{aligned} \quad (21)$$

where the series is convergent by Lemma 3.1, and its sum is  $H_{f,k}(r+1, \dots, r+1)$ .

Let  $I$  be fixed and assume that  $I = \{1, 2, \dots, s\}$ , that is  $d_1, \dots, d_s > x$  and  $d_{s+1}, \dots, d_k \leq x$ , where  $s \geq 1$ . We deduce, by noting that  $A - (r+1) = -\frac{1}{2} \min(r+1, 1) < 0$ ,

$$\begin{aligned} & \sum_{\substack{d_1, \dots, d_s > x \\ d_{s+1}, \dots, d_k \leq x}} \frac{|h_{f,k}(d_1, \dots, d_k)|}{d_1^{r+1} \dots d_k^{r+1}} \\ &= \sum_{\substack{d_1, \dots, d_s > x \\ d_{s+1}, \dots, d_k \leq x}} \frac{|h_{f,k}(d_1, \dots, d_k)| d_1^{A-(r+1)+\varepsilon} \dots d_s^{A-(r+1)+\varepsilon}}{d_1^{A+\varepsilon} \dots d_s^{A+\varepsilon} d_{s+1}^{r+1} \dots d_k^{r+1}} \\ &\leq x^{s(A-(r+1)+\varepsilon)} \sum_{d_1, \dots, d_k=1}^{\infty} \frac{|h_{f,k}(d_1, \dots, d_k)|}{d_1^{A+\varepsilon} \dots d_s^{A+\varepsilon} d_{s+1}^{r+1} \dots d_k^{r+1}} \\ &= x^{s(A-(r+1)+\varepsilon)} H_{f,k}(A+\varepsilon, \dots, A+\varepsilon, r+1, \dots, r+1) \\ &\ll x^{-\frac{s}{2}\min(r+1,1)+s\varepsilon}, \end{aligned}$$

the latter series (the same as before) being convergent, and the obtained error is maximal for  $s = 1$  giving, according to (19) and (21), the same error

$$O\left(x^{k(r+1)-\frac{1}{2}\min(r+1,1)+\varepsilon}\right).$$

This proves asymptotic formula (7) with the constant  $C_{f,k} = H_{f,k}(r+1, \dots, r+1)$ . Here, according to Lemma 3.1,

$$C_{f,k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1, \dots, \nu_k=0}^{\infty} \frac{f(p^{\max(\nu_1, \dots, \nu_k)})}{p^{(r+1)(\nu_1 + \dots + \nu_k)}}.$$

The proof of (8) is similar, based on Lemma 3.1 and the convolutional identity

$$\frac{f([n_1, \dots, n_k])}{(n_1 \dots n_k)^r} = \sum_{j_1 d_1 = n_1, \dots, j_k d_k = n_k} \frac{h_{f,k}(d_1, \dots, d_k)}{d_1^r \dots d_k^r},$$

which implies that

$$\sum_{n_1, \dots, n_k \leq x} \frac{f([n_1, \dots, n_k])}{(n_1 \dots n_k)^r} = \sum_{d_1, \dots, d_k \leq x} \frac{h_{f,k}(d_1, \dots, d_k)}{d_1^r \dots d_k^r} \sum_{j_1 \leq x/d_1} 1 \dots \sum_{j_k \leq x/d_k} 1.$$

□

*Proof of Theorem 2.2.* Formula (9) is obtained by using Lemma 3.2, in exactly the same way as (7) (here  $r \geq 0$  and  $R = \max(r, 0) = r$ ), with the constant  $D_{f,k} = \overline{H}_{f,k}(r+1, \dots, r+1)$ .  $\square$

*Proof of Corollary 1.* Apply Theorem 2.1 for  $f = \text{id}_r$ . Here

$$\begin{aligned} C_{r,3} &= \prod_p \left(1 - \frac{1}{p}\right)^3 \sum_{a,b,c=0}^{\infty} \frac{p^{r \max(a,b,c)}}{p^{(r+1)(a+b+c)}} \\ &= \prod_p \left(1 - \frac{1}{p}\right)^3 (6S_1 + 3S_2 + 3S_3 + S_4), \end{aligned}$$

with

$$\begin{aligned} S_1 &= \sum_{0 \leq a < b < c} \frac{p^{rc}}{p^{(r+1)(a+b+c)}}, & S_2 &= \sum_{0 \leq a=b < c} \frac{p^{rc}}{p^{(r+1)(2a+c)}}, \\ S_3 &= \sum_{0 \leq a < b=c} \frac{p^{rc}}{p^{(r+1)(a+2c)}}, & S_4 &= \sum_{0 \leq a=b=c} \frac{p^{rc}}{p^{(r+1)3c}}, \end{aligned}$$

which gives (11). Formula (12) for the constant  $C_{r,4}$  can be computed in a similar manner.  $\square$

*Proof of Corollary 2.* Apply Theorem 2.2 for  $f = \text{id}_r$ . The constants  $D_{r,3}$  and  $D_{r,4}$  can be evaluated like above.  $\square$

*Proof of Corollaries 3, 4, 5.* Apply Theorem 2.1 for  $f = \sigma$ ,  $f = \varphi$  with  $r = 1$ , resp.  $f = \mu^2$  with  $r = 0$ .  $\square$

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